# Portals and Complex Analysis

By **finegeometer**.

## Contents



## Theory

Consider portal spaces in two dimensions, and restrict to those with a well-defined sense of distance and direction. That is, those whose portals do not rotate or scale you.

Such spaces are *exactly characterized* by pairs $(M, f)$, where $M$ is a compact Riemann surface and $f$ is a meromorphic covector field.

$M$ describes the points of the portal space, including portal boundaries and "points at infinity". $f$ describes distances and directions; if you travel along the path $\gamma : [0, 1] \to M$, your total displacement is the integral $\int_\gamma f(z)\,\mathrm{d}z$.

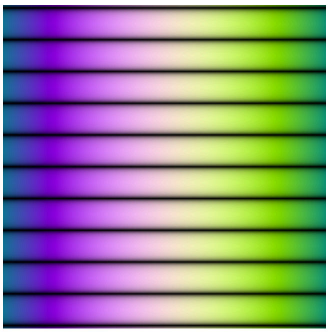
Any other covector field $g$ can be interpreted as a gravitational field, since the conditions for complex differentiability force it to be irrotational and divergence-free. The corresponding potential is the negative real part of $g$'s integral, so if we want conservation of energy, we should require the integral of $g$ around any loop to be purely imaginary.

## Examples

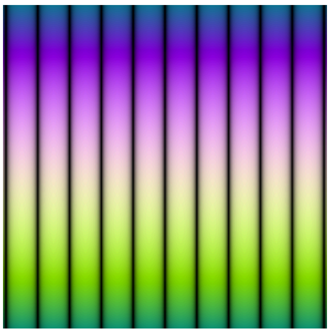
### One Universe, No Portal

Let's start simple, with the portal space $(M, f) = \left(\hat{\mathbb{C}}, z \mapsto 1\right)$. This describes a universe without portals, in the typical coordinate system. After all, the displacement from point 0 to point $a$ is $\int_0^a f(z)\,\mathrm{d}z = \int_0^a \mathrm{d}z = a$, so $a$ simply acts as a coordinate.

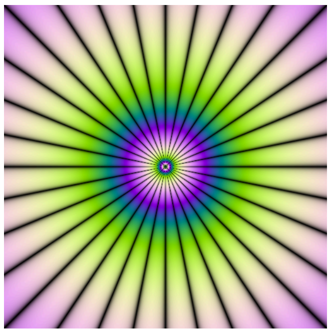
As for gravity, we have many choices. Some realistic; some less so.



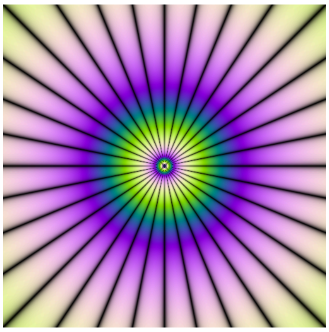
Rightwards Gravity
$g(z) = 1$



Downwards Gravity
$g(z) = i$



Point Mass
$g(z) = -\frac{1}{z}$

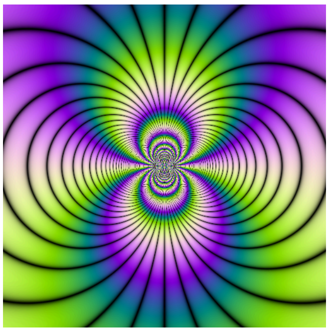


Repulsive Gravitational Source

$g(z) = \frac{1}{z}$

Gravitational Vortex
(violates energy conservation)

$g(z) = \frac{i}{z}$



Gravitational Dipole

$g(z) = \frac{1}{z^2}$

## Two Universes, Infinite Portal

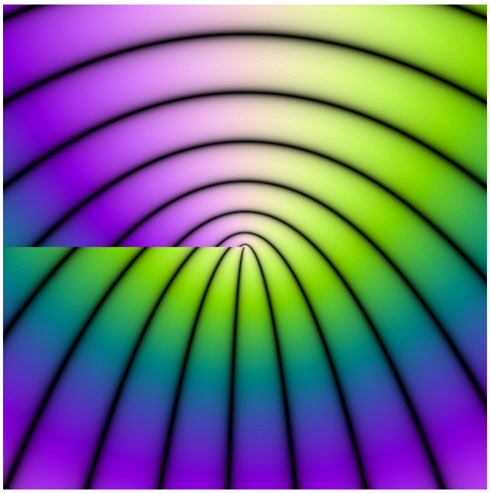
Okay, let's try the next simplest possibility: $(M, \mathbb{C}) \in \left(\hat{\mathbb{C}}, z \mapsto z\right)$. The displacement from point 0 to point $a$ is $\int_0^a f(z)\,\mathrm{d}z = \int_0^a z\,\mathrm{d}z = \frac{a^2}{2}$.

But that means points $a$ and $-a$ correspond to the same physical location. In effect, we have two connected universes.

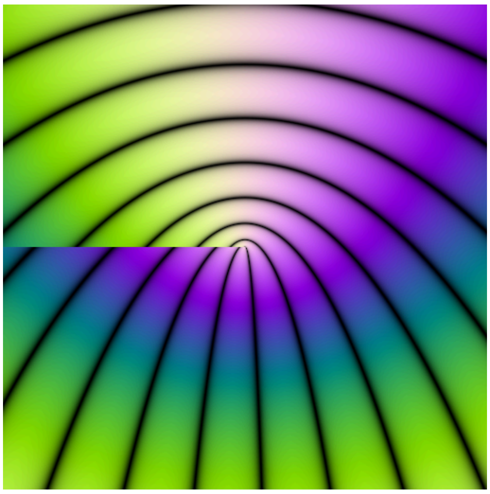
But how are they connected?

Imagine following the path $t \mapsto re^{it}$, as $t$ ranges from 0 to $\pi$. Our physical position is then given by $\frac{(re^{it})^2}{2} = \frac{r^2}{2}e^{2it}$, so we're walking in a circle of radius $\frac{r^2}{2}$ around the origin. But since $re^{i\pi} \neq re^{i0}$, we don't end up where we started. Therefore, walking in a circle around the origin always takes you from one universe to the other.

So there's a portal boundary at the origin. And the portal surface must stretch all the way out to infinity, since even a very large circle takes you to the other universe.



First Universe



Second Universe

# Poles and Zeroes

It's a bit annoying that the portal surface is infinite. Why did that happen?

This is a question about the nature of the portal space at infinity. Points at infinity correspond to *poles* of $f$, so we should analyze the nature of these poles.

Annoyingly, in our examples, these poles are at $z = \infty$. There's such a thing as a "pole at infinity", but its definition seems designed specifically for scalar fields, not covector fields.

So we should change coordinates. Applying the transformation $z \mapsto \frac{1}{z} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and recalling how covector fields transform, we have:

- $\left(\hat{\mathbb{C}}, f\right) \cong \left(\hat{\mathbb{C}}, z \mapsto -\frac{f(\frac{1}{z})}{z^2}\right)$
- $\left(\hat{\mathbb{C}}, z \mapsto 1\right) \cong \left(\hat{\mathbb{C}}, z \mapsto -\frac{1}{z^2}\right)$
- $\left(\hat{\mathbb{C}}, z \mapsto z\right) \cong \left(\hat{\mathbb{C}}, z \mapsto -\frac{1}{z^3}\right)$

We see that the no-portal space has a *second-order* pole, while the space with a portal of infinite length has a *third-order* pole.

In general, if $f$ has an $n$th order pole, then walking in a large circle takes you through $n - 1$ different universes before you return to your starting point. (The $n = 1$ case looks like a universe rolled into a cylinder.) So if you want to avoid infinite portals, you should aim for every pole to be second-order.

In addition to looking at $f$'s poles, we should also look at its zeroes.

When $f(a)$ is a pole of order $n$, for $a \neq \infty$, the displacement $\int_a^z f(z)\,\mathrm{d}z$ has a zero of order $n + 1$. So as $z$ closely circles $a$, the coordinate of $z$ makes $n + 1$ small circles around the coordinate of $a$. This means zeroes of $f$ denote portal boundaries, and the order of the zero tells you the nature of the portal boundary.

If you know the portal structure you are aiming for, you have information about the poles and zeroes of $f$, and the topology of $M$. This information nearly determines the pair $(M, f)$, leaving only a finite-dimensional parameter space of possibilities.

## More Examples

### Two Universes, Finite Portal

Suppose we want a finite-size portal between two otherwise disconnected universes. We compute:

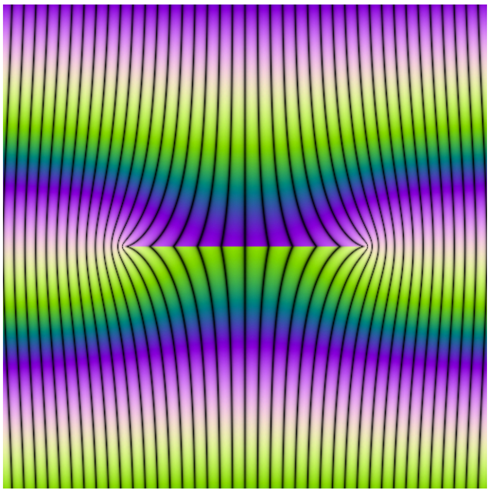
- The topology of this space is a sphere, once the points at infinity are filled in. So choose $M = \hat{\mathbb{C}}$.
- There are two points at infinity, one for each universe, and each is portal-free. So $f$ will have two second-order poles.
- There should be two portal boundaries. (You might have expected four: two for each side of the portal. But since the portal connects the space, you've really listed two points twice each.) So $f$ should have two first-order zeroes.
- This is consistent with the Poincare-Hopf index theorem, which implies that the sum of the orders of the zeroes (with poles counting negative) should be the negative Euler characteristic of $M$.

This is easy to achieve:

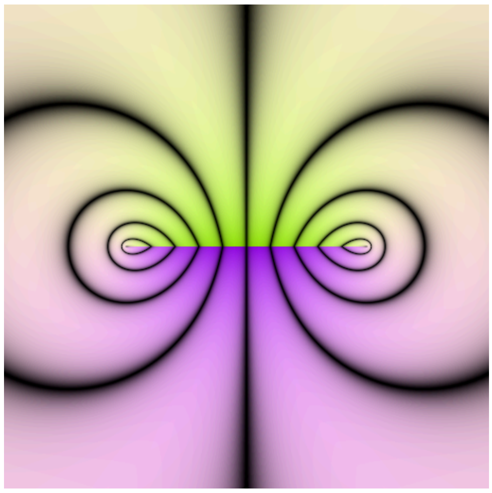
$$(M, f) = \left(\hat{\mathbb{C}}, z \mapsto 1 - \frac{1}{z^2}\right)$$

Poles at 0 and $\infty$; zeroes at $-1$ and 1.

Picking a simple gravitational field, $g = z \mapsto i$, we get the following picture.



First Universe



Second Universe

But how would we calculate the gravitational field in coordinate terms? If we wanted to understand how it grows at the portal boundary?

The coordinate position corresponding to $z$ is computed as $\int_0^z f(z')\,\mathrm{d}z' = z + \frac{1}{z}$. Inverting this:

$$z + \frac{1}{z} = w$$
$$z^2 - wz + 1 = 0$$
$$z = \frac{w \pm \sqrt{w^2 - 4}}{2}$$

So coordinate position $w$ comes from $z = \frac{w \pm \sqrt{w^2-4}}{2}$, with the $\pm$ choosing which universe we're in.

We further compute:

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{1}{\frac{\mathrm{d}w}{\mathrm{d}z}} = \frac{1}{f(z)} = \frac{1}{1 - z^2}$$
$$= \frac{1}{1 - \left(\frac{w \pm \sqrt{w^2-4}}{2}\right)^2}$$

So the gravitational field, in terms of the coordinate position $w$, is

$$\left(\frac{1}{1 - \left(\frac{w \pm \sqrt{w^2-4}}{2}\right)^2}\right) g(w) = \frac{i}{1 - \left(\frac{w \pm \sqrt{w^2-4}}{2}\right)^2}$$

Near a portal boundary, say $w = 2 + \varepsilon$ with $\varepsilon \ll 1$, this becomes:

$$\frac{i}{1 - \left(\frac{(2+\varepsilon) \pm \sqrt{(2+\varepsilon)^2 - 4}}{2}\right)^2}$$
$$= \frac{i}{1 - \left(1 + \frac{\varepsilon}{2} \pm \sqrt{\varepsilon + \frac{\varepsilon^2}{4}}\right)^2}$$
$$= \frac{i}{1 - (1 + \sqrt{\varepsilon} + O(\varepsilon))^2}$$

$$
\begin{aligned}
&= \frac{i}{1 - (1 + 2\sqrt{\varepsilon} + O(\varepsilon))} \\
&= \frac{i}{2\sqrt{\varepsilon} + O(\varepsilon)} \\
&= \frac{i}{2\sqrt{\varepsilon}} + O(1)
\end{aligned}
$$

So as we approach the portal boundary, the field strength grows as the inverse square root of the distance from the portal boundary. It's unbounded, but the growth rate is slower than the field of a point particle.

## One Universe, Finite Portals

We'll now try something harder: a portal between two parts of the same universe.

For the first time, we do not have $M = \hat{\mathbb{C}}$. Our portal space is topologically a *torus*, not a sphere.

Fortunately, complex toruses have a simple classification; they're always the quotient of $\mathbb{C}$ by some lattice. And we can take that lattice to be generated by 1 and $\tau$, where $\tau$ has positive imaginary part.

We have interesting topology! There are two independent ways to loop around the torus: add 1, and add $\tau$. And there are two independent ways to loop around the portal space: loop *through* the portals, or circle *around* one end of the portal. Let's say looping through the portals corresponds to the $\tau$ direction, while looping around a portal end corresponds to the 1 direction.

Then we want to choose $f$ to be a covector field on the torus $\mathbb{C}/\langle 1, \tau \rangle$, such that $\int_0^1 f(z)\,\mathrm{d}z = 0$, but $\int_0^\tau f(z)\,\mathrm{d}z \neq 0$. We also want a single pole of order two, and two zeroes of order one.

What function might have these properties? A Jacobi theta function comes close.

$$
\begin{aligned}
\vartheta(z;\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i n z} \\
\vartheta(z+1;\tau) &= \vartheta(z;\tau) \\
\vartheta(z+\tau;\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i n (z+\tau)} \\
&= \sum_{n\in\mathbb{Z}} e^{\pi i \tau (n^2+2n) + 2\pi i n z} \\
&= \sum_{n\in\mathbb{Z}} e^{\pi i \tau (n^2+2n+1) + 2\pi i (n+1) z} e^{-\pi i (\tau + 2z)} \\
&= \sum_{m\in\mathbb{Z}} e^{\pi i \tau m^2 + 2\pi i m z} e^{-\pi i (\tau + 2z)} \\
&= \vartheta(z+\tau;\tau) e^{-\pi i (\tau + 2z)}
\end{aligned}
$$

Periodic in the 1 direction and quasiperiodic in the $\tau$ direction, but it's not quite right.

$$
\ln(\vartheta(z+\tau;\tau)) = \ln(\vartheta(z;\tau)) - \pi i (2z + \tau)
$$

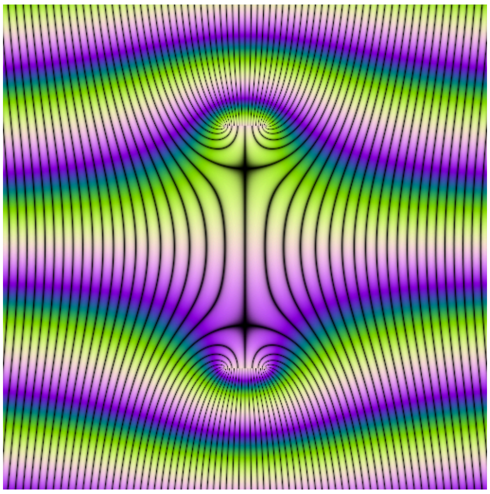
Closer…

$$
\frac{\mathrm{d}}{\mathrm{d}z} \ln(\vartheta(z+\tau;\tau)) = \frac{\mathrm{d}}{\mathrm{d}z} \ln(\vartheta(z;\tau)) - 2\pi i
$$

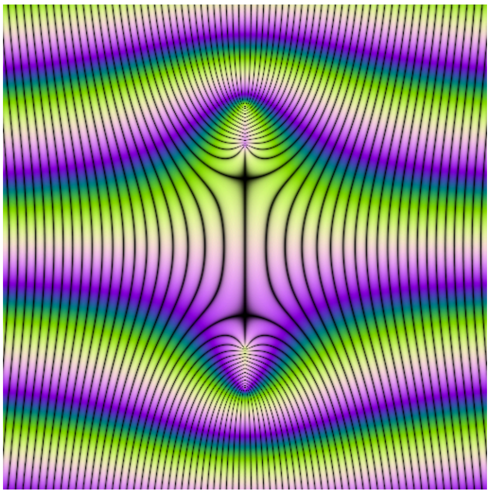
There. That'll work for $\int f(z)\,\mathrm{d}z$.

$$(M, f) = \left(\mathbb{C}/\langle 1, \tau\rangle, z \mapsto \frac{\mathrm{d}^2}{\mathrm{d}z^2} \ln(\vartheta(z + \tau; \tau))\right)$$

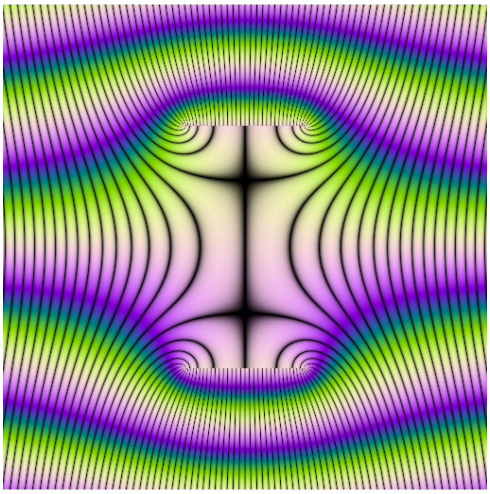
This portal always teleports you by a distance of $2\pi$, vertically. The size and angle of the portal are controlled by $\tau$. To find a useful gravitational field, set $g(z) = if(z) + C$, where $C$ is whatever constant simultaneously makes $\int_0^1 g(z)\,\mathrm{d}z$ and $\int_0^\tau g(z)\,\mathrm{d}z$ be pure imaginary.



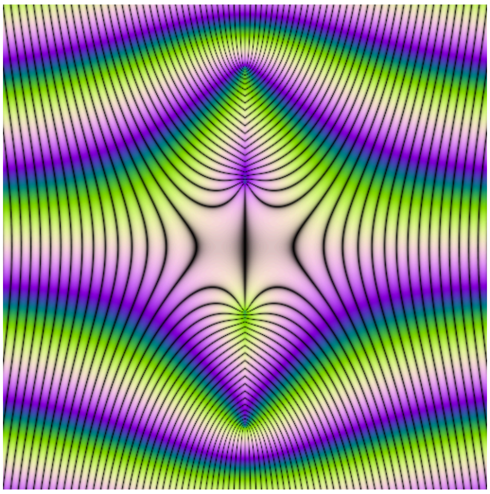
$\tau = i$



$\tau = \frac{1}{2} + i$



$\tau = \frac{2i}{3}$



$\tau = \frac{1}{2} + \frac{2i}{3}$

## Technical Aspects

How am I making these pictures?

We have a coordinate map $\int f : M \to \mathbb{C}$, and a potential map $-\int g : M \to \mathbb{C}$. For each pixel $w$ on the screen, we'd like to compute $\left(\int g\right)\left(\left(\int f\right)^{-1}(w)\right)$. But that's awkward — I don't want to invert a theta function!

Instead, I start with a mesh, whose vertices' are equipped with the $M$-points they correspond to. In the vertex shader, I apply $\int f$ to compute the corresponding physical coordinates. But I also pass the $M$-point to Then the fragment shader, where I can simply apply $\int g$. No inverse functions needed!

As stated, the above works if the initial mesh is sufficiently fine. But if it's coarse, the interpolated $M$-point that the fragment shader gets isn't quite accurate. Instead, I treat it as a good *estimate* of $\left(\int f\right)^{-1}(w)$, and apply one iteration of the Newton-Raphson method to improve the accuracy.